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# *Differential Equations Admitting a Given Group.*

By J. Edmund Wright.

Let there be given a continuous $r$-parameter group in the $n$ variables $x_1, x_2, \ldots, x_{n-1}, y$, and let the $x$'s be functions of $y$. Suppose a system of $n-1$ differential equations is given by $I_1 = 0,\ I_2 = 0, \ldots, I_{n-1} = 0$, where the $I$'s are differential invariants of the group, of orders $a_1, a_2, \ldots, a_{n-1}$ respectively. Then their complete solution involves $N$ arbitrary constants. But since the system is invariant under the given group, it follows that from a solution involving $N-r$ constants we can deduce the general solution, provided the particular solution is not invariant under any subgroup of the given group.

The general solution consists of a set of curves, $\infty^N$ in number, in space of $n$ dimensions, and these curves are merely interchanged under the operations of the group, so that, *e. g.*, from a single curve we may derive by means of the transformations of the group a set of $\infty^r$ curves, provided that the single curve is not invariant under any transformation of the group. All of these are solutions of the differential equations.

Now any curve may be expressed by equations of the type $x_i = f_i(t)$, where $f$ is an appropriate function of $t$, and $t$ is parametric. If the group be extended by adding the variable $t$, and assuming $x_1, x_2, \ldots, x_{n-1}, y$ to be functions of $t$, whilst $t$ is an invariant, we get $n-1$ equations in $n$ dependent variables, all of which are invariants of the group as thus modified.

In cases that frequently arise, *e.g.*, in differential geometry or in dynamics, the independent variable is an invariant of the group, and may itself be taken as $t$. In other cases we change the independent variable as indicated, and we then need an equation defining $t$. This equation must be one among the $n$ dependent variables and $t$, and if it is quite general it may involve derivatives of the dependent variables. Suppose it to be solved for $t$; then since $t$ is an invariant of the group, if the new system is to remain an invariant system, we must have a relation of the form $I_n = t$, where $I_n$ is an invariant of the group. For the rest we may choose $I_n$ arbitrarily except that it must be independent of $I_1, I_2, \ldots, I_{n-1}$. We have finally $n$ equations for the $n$ dependent variables as functions of $t$. Their solution depends on $N'$ constants, where $N'$ may be greater than $N$, but this gives not only the integral curves of the original system, but also the parametric representation of these integral curves in terms of a particular parameter $t$. This accounts for the additional constants involved.

Now if $I$ be any invariant of the group, $\frac{dI}{dt}$ is also an invariant. If we regard $\frac{dI}{dt}$ as depending on $I$, then it may be proved that the group contains precisely $n$ independent invariants in addition to $t$. We indicate the proof if $r < n$. Suppose that $r - h$ of the operators of zero order are unconnected, $r - h_1$ of the operators of unit order unconnected and so on. Then $h \geq h_1 \geq h_2 \ldots$ and, finally, for some value of $\lambda$, $h_\lambda$ is zero. We have $n - r + h$ invariants of zero order, $2n - r + h_1$ of order $\leq 1$, and so on. We thus have $pn - r + h_p$ invariants of order $\leq p$ and $(p-1)n - r + h_{p-1}$ invariants of order $\leq p - 1$. Thus there are precisely $n - (h_{p-1} - h_p)$ invariants of order $p$. Now there are $n - r + h$ invariants of zero order, and therefore this number of invariants of unit order are of the form $\frac{dI}{dt}$. Thus the number of new first-order invariants is

$$n - h + h_1 - (n - r + h) = r + h_1 - 2h.$$

The number of new $p$-th order invariants is

$$n - (h_{p-1} - h_p) - [n - (h_{p-2} - h_{p-1})] = h_{p-2} + h_p - 2h_{p-1}.$$

Thus, finally, the total number of independent invariants is

$$(n - r + h) + (r + h_1 - 2h) + (h + h_2 - 2h_1) + \ldots = n.$$

Let these independent invariants be $X_1, X_2, \ldots, X_n$; then the solution of the differential equations $X_i = F_i(t)$ in the original variables will contain $r$ arbitrary constants. Further, any system of equations of the type $I = 0$ may be expressed in terms of $t, X_1, \ldots, X_n, \ldots, \frac{d^i X_k}{dt^i}, \ldots$. Thus the original system may be integrated in two steps. First we have a set of $n$ equations in the $n$ variables $X$, and $t$. If these are integrated they give solutions

$$X_1 = F_1(t),\ X_2 = F_2(t),\ \ldots,\ X_n = F_n(t),$$

which involve $N'$ constants. Now, regarding the $X$'s as functions of the $x$'s, $y$ and their derivatives, we have to integrate a system of invariant equations of order $r$. But the solutions of this last system are the different curves derived from a single curve by the operations of the group. We therefore need to know a single solution to determine the complete set, provided that the single solution is not invariant under any operation of the group.

For example, consider the group of rotations about the origin in space of three dimensions; the differential operators are

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

and if $t$ is the independent variable the invariants of the group are

$$\Sigma x^2, \ \Sigma x'^2, \ \Sigma x''^2, \ \ldots, \ \Sigma xx', \ \Sigma xx'', \ \ldots,$$

$$\Delta \equiv \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}, \text{ etc.}$$

We may choose as our invariants $X_1$, $X_2$, $X_3$,

$$X_1 = \Sigma x^2, \ X_2 = \Sigma x'^2, \ X_3 = \Delta,$$

and *then any system of equations invariant under the group will involve for their final solution the determination of a single solution of*

$$X_1 = f_1(t), \ X_2 = f_2(t), \ X_3 = f_3(t).$$

We now take $x\sqrt{f_1(t)}$ instead of $x$, $y\sqrt{f_1(t)}$ instead of $y$, and $z\sqrt{f_1(t)}$ instead of $z$, and introduce instead of $t$ a new variable $t_1$ defined by

$$dt_1 = \sqrt{\frac{f_2(t)}{f_1(t) - \frac{1}{2}f_1'(t)/\sqrt{f_1(t)}}}\,dt.$$

With these changes our system of equations becomes

$$\Sigma x^2 = 1, \ \Sigma x'^2 = 1, \ \Delta = f(t),$$

where we now drop the suffix from $t_1$; and we shall assume that $f$ is not zero. We see that $\Sigma xx' = 0$, hence $x$, $y$, $z$, and $x'$, $y'$, $z'$, are the direction cosines of two straight lines at right angles. Also

$$\Sigma xx'' = -\Sigma x'^2 = -1.$$

Thus

$$\Sigma x(x'' + x) = 0, \ \Sigma x'(x'' + x) = 0,$$

and therefore the line perpendicular to the two given lines must have its direction cosines proportional to $x'' + x$, etc. It follows from the third equation that these direction cosines are precisely $(x'' + x)/f(t)$, etc. From the known relations that exist among the direction cosines of three mutually perpendicular straight lines we have at once

$$\left(\frac{x'' + x}{f}\right)^2 + x'^2 + x^2 = 1, \tag{1}$$

a differential equation of the second order for $x$. If this be differentiated, it gives the linear equation of the third order

$$x''' - \frac{f'}{f} x'' + (1 + f^2)x' - \frac{f'}{f} x = 0; \tag{2}$$

(it may easily be seen that the apparent solution $x'' + x = 0$ must be excluded). We determine any particular solution of (2) that satisfies (1), and then by elimination we get a first-order equation for $y$. If we can find any particular solution of the equation for $y$, $z$ is determinate, and then the complete solution of the set of equations may be at once written down.

As an illustration, suppose that $x = kt$ is a particular solution of (1); then $f = kt/(1 - k^2 - k^2t^2)^{\frac{1}{2}}$. Corresponding particular values of $y$ and $z$ are given by

$$\begin{aligned} f(1 - k^2)^{\frac{1}{2}} y &= kt\,(\cos u + kf \sin u), \\ f(1 - k^2)^{\frac{1}{2}} z &= kt\,(\sin u - kf \cos u), \end{aligned}$$

where $\sin ku = kt/(1 - k^2)^{\frac{1}{2}}$.

If we call these particular values $x_0$, $y_0$, $z_0$, the general solution of the equations is

$$\begin{aligned} x &= l\,x_0 + m\,y_0 + n\,z_0, \\ y &= l'x_0 + m'y_0 + n'z_0, \\ z &= l''x_0 + m''y_0 + n''z_0, \end{aligned}$$

where the constants $l$, $m$, $n$, etc. are the direction cosines of three mutually perpendicular straight lines.

BRYN MAWR COLLEGE, *November*, 1908.